

**POSITIVE INTEGER SOLUTIONS OF THE PELL
EQUATION $x^2 - dy^2 = N$, $d \in \{k^2 \pm 4, k^2 \pm 1\}$ AND $N \in \{\pm 1, \pm 4\}$**

REFİK KESKİN AND MERVE GÜNEY

ABSTRACT. Let k be a natural number and $d = k^2 \pm 4$ or $k^2 \pm 1$. In this paper, by using continued fraction expansion of \sqrt{d} , we find fundamental solution of the equations $x^2 - dy^2 = \pm 1$ and we get all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$ in terms of generalized Fibonacci and Lucas sequences. Moreover, we find all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ in terms of generalized Fibonacci and Lucas sequences. Although some of the results are well known, we think our method is elementary and different from the others.

1. INTRODUCTION

Let d be a positive integer that is not a perfect square. It is well known that the Pell equation $x^2 - dy^2 = 1$ has always positive integer solutions. When $N \neq 1$, the Pell equation $x^2 - dy^2 = N$ may not have any positive integer solutions. It can be seen that the equations $x^2 - 3y^2 = -1$ and $x^2 - 7y^2 = -4$ have no positive integer solutions. Whether or not there exists a positive integer solution to the equation $x^2 - dy^2 = -1$ depends on the period length of the continued fraction expansion of \sqrt{d} (See section 2 for more detailed information). When k is a positive integer and $d \in \{k^2 \pm 4, k^2 \pm 1\}$, positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ and $x^2 - dy^2 = \pm 1$ have been investigated by Jones in [6] and the method used in the proofs of the theorems are method of descent of Fermat. The same or similar equations are investigated by some other authors in [9], [10], [17], [8], and [16]. Especially, when a solution exists, all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ and $x^2 - dy^2 = \pm 1$ are given in terms of the generalized Fibonacci and Lucas sequences. In this paper, if a solution exists, we will use continued fraction expansion of \sqrt{d} in order to get all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$ when $d \in \{k^2 \pm 4, k^2 \pm 1\}$. Moreover, we will find all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ when $d \in \{k^2 \pm 4, k^2 \pm 1\}$.

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Our proofs are elementary and we think our method is new and different from the others.

Now we briefly mention the generalized Fibonacci and Lucas sequences $(U_n(k, s))$ and $(V_n(k, s))$. Let k and s be two nonzero integers with $k^2 + 4s > 0$. Generalized Fibonacci sequence is defined by

$$U_0(k, s) = 0, U_1(k, s) = 1 \text{ and } U_{n+1}(k, s) = kU_n(k, s) + sU_{n-1}(k, s)$$

for $n \geq 1$ and generalized Lucas sequence is defined by

$$V_0(k, s) = 2, V_1(k, s) = k \text{ and } V_{n+1}(k, s) = kV_n(k, s) + sV_{n-1}(k, s)$$

for $n \geq 1$, respectively. For $k = s = 1$, the sequences (U_n) and (V_n) are called Fibonacci and Lucas sequences and they are denoted as (F_n) and (L_n) , respectively. For $k = 2$ and $s = 1$, the sequences (U_n) and (V_n) are called Pell and Pell-Lucas sequences and they are denoted as (P_n) and (Q_n) , respectively. It is well known that

$$U_n(k, s) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n(k, s) = \alpha^n + \beta^n$$

where $\alpha = \left(k + \sqrt{k^2 + 4s}\right) / 2$ and $\beta = \left(k - \sqrt{k^2 + 4s}\right) / 2$. The above identities are known as Binet's formulae. Clearly, $\alpha + \beta = k$, $\alpha - \beta = \sqrt{k^2 + 4s}$, and $\alpha\beta = -s$. Especially, if $\alpha = \left(k + \sqrt{k^2 + 4}\right) / 2$ and $\beta = \left(k - \sqrt{k^2 + 4}\right) / 2$, then we get

$$U_n(k, 1) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n(k, 1) = \alpha^n + \beta^n. \quad (1.1)$$

If $\alpha = \left(k + \sqrt{k^2 - 4}\right) / 2$ and $\beta = \left(k - \sqrt{k^2 - 4}\right) / 2$, then we get

$$U_n(k, -1) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n(k, -1) = \alpha^n + \beta^n. \quad (1.2)$$

Also, if $\alpha = (1 + \sqrt{5}) / 2$ and $\beta = (1 - \sqrt{5}) / 2$, then we get

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n. \quad (1.3)$$

Moreover, if k is even, then it can be easily seen that

$$\begin{aligned} U_n(k, \pm 1) \text{ is odd} &\Leftrightarrow n \text{ is odd,} \\ U_n(k, \pm 1) \text{ is even} &\Leftrightarrow n \text{ is even,} \\ V_n(k, \pm 1) \text{ is even} &\text{ for all } n \in \mathbb{N}. \end{aligned} \quad (1.4)$$

If k is odd, then

$$2 \mid V_n(k, \pm 1) \Leftrightarrow 2 \mid U_n(k, \pm 1) \Leftrightarrow 3 \mid n.$$

For more information about generalized Fibonacci and Lucas sequences, one can consult [14], [7], [13], [9], and [10].

2. PRELIMINARIES

Let d be a positive integer which is not a perfect square and N be any nonzero fixed integer. Then the equation $x^2 - dy^2 = N$ is known as Pell equation. For $N = \pm 1$, the equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$ are known as classical Pell equation. If $a^2 - db^2 = N$, we say that (a, b) is a solution to the Pell equation $x^2 - dy^2 = N$. We use the notations (a, b) and $a + b\sqrt{d}$ interchangeably to denote solutions of the equation $x^2 - dy^2 = N$. Also, if a and b are both positive, we say that $a + b\sqrt{d}$ is positive solution to the equation $x^2 - dy^2 = N$. Continued fraction plays an important role in solutions of the Pell equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$. Let d be a positive integer that is not a perfect square. Then there is a continued fraction expansion of \sqrt{d} such that $\sqrt{d} = [a_0, \overline{a_1, a_2, \dots, a_{l-1}, 2a_0}]$ where l is the period length and the a_j 's are given by the recursion formula;

$$\alpha_0 = \sqrt{d}, \quad a_k = \lfloor \alpha_k \rfloor \quad \text{and} \quad \alpha_{k+1} = \frac{1}{\alpha_k - a_k}, \quad k = 0, 1, 2, 3, \dots$$

Recall that $a_l = 2a_0$ and $a_{l+k} = a_k$ for $k \geq 1$. The n^{th} convergent of \sqrt{d} for $n \geq 0$ is given by

$$\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}$$

Let $x_1 + y_1\sqrt{d}$ be a positive solution to the equation $x^2 - dy^2 = N$. We say that $x_1 + y_1\sqrt{d}$ is the fundamental solution of the equation $x^2 - dy^2 = N$, if $x_2 + y_2\sqrt{d}$ is different solution to the equation $x^2 - dy^2 = N$, then $x_1 + y_1\sqrt{d} < x_2 + y_2\sqrt{d}$. Recall that if $a + b\sqrt{d}$ and $r + s\sqrt{d}$ are two solutions to the equation $x^2 - dy^2 = N$, then $a = r$ if and only if $b = s$, and $a + b\sqrt{d} < r + s\sqrt{d}$ if and only if $a < r$ and $b < s$. The following lemmas and theorems can be found in many elementary textbooks.

Lemma 1. *If $x_1 + y_1\sqrt{d}$ is the fundamental solution to the equation $x^2 - dy^2 = -1$, then $(x_1 + y_1\sqrt{d})^2$ is the fundamental solution to the equation $x^2 - dy^2 = 1$.*

If we know fundamental solution of the equations $x^2 - dy^2 = \pm 1$ and $x^2 - dy^2 = \pm 4$, then we can give all positive integer solutions to these equations. Our theorems are as follows. For more information about Pell equation, one can consult [12], [15], and [4]. Now we give the fundamental solution of the

equations $x^2 - dy^2 = \pm 1$ by means of the period length of the continued fraction expansion of \sqrt{d} .

Lemma 2. *Let l be the period length of continued fraction expansion of \sqrt{d} . If l is even, then the fundamental solution to the equation $x^2 - dy^2 = 1$ is given by*

$$x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}$$

and the equation $x^2 - dy^2 = -1$ has no integer solutions. If l is odd, then the fundamental solution of the equation $x^2 - dy^2 = 1$ is given by

$$x_1 + y_1\sqrt{d} = p_{2l-1} + q_{2l-1}\sqrt{d}.$$

and the fundamental solution to the equation $x^2 - dy^2 = -1$ is given by

$$x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}.$$

Theorem 1. *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = 1$. Then all positive integer solutions to the equation $x^2 - dy^2 = 1$ are given by*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

with $n \geq 1$.

Theorem 2. *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = -1$. Then all positive integer solutions to the equation $x^2 - dy^2 = -1$ are given by*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^{2n-1}$$

with $n \geq 1$. Now we give the following two theorems from [15]. See also [4].

Theorem 3. *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 - dy^2 = 4$. Then all positive integer solutions to the equation $x^2 - dy^2 = 4$ are given by*

$$x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^n}{2^{n-1}}$$

with $n \geq 1$.

Theorem 4. Let $x_1 + y_1\sqrt{d}$ be the fundamental solution to the equation $x^2 -$

$dy^2 = -4$. Then all positive integer solutions to the equation $x^2 - dy^2 = -4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^{2n-1}}{4^{n-1}}$$

with $n \geq 1$.

From now on, we will assume that k is a natural number. We give continued fraction expansion of \sqrt{d} for $d = k^2 \pm 4$. The proofs of the following two theorems are easy and they can be found many text books on number theory as an exercise (see, for example [2]).

Theorem 5. Let $k > 1$. Then

$$\sqrt{k^2 + 4} = \begin{cases} \left[k, \frac{k}{2}, 2k \right] & \text{if } k \text{ is even,} \\ \left[k, \frac{k-1}{2}, 1, 1, \frac{k-1}{2}, 2k \right] & \text{if } k \text{ is odd.} \end{cases}$$

Theorem 6. Let $k > 3$. Then

$$\sqrt{k^2 - 4} = \begin{cases} \left[k-1, 1, \frac{k-3}{2}, 2, \frac{k-3}{2}, 1, 2(k-1) \right] & \text{if } k \text{ is odd,} \\ \left[k-1, 1, \frac{k-4}{2}, 1, 2(k-1) \right] & \text{if } k \text{ is even and } k \neq 4 \\ [3, \overline{2}, 6] & \text{if } k = 4 \end{cases}$$

Corollary 1. Let $k > 1$ and $d = k^2 + 4$. If k is odd, then the fundamental solution to the equation $x^2 - dy^2 = -1$ is

$$x_1 + y_1\sqrt{d} = \frac{k^3 + 3k}{2} + \frac{k^2 + 1}{2}\sqrt{d}.$$

If k is even, the equation $x^2 - dy^2 = -1$ has no positive integer solutions.

Proof. Assume that k is odd. Then the period length of the continued fraction expansion of $\sqrt{k^2 + 4}$ is 5 by Theorem 5. Therefore the fundamental solution of the equation $x^2 - dy^2 = -1$ is $p_4 + q_4\sqrt{d}$ by Lemma 2. Since

$$\frac{p_4}{q_4} = k + \frac{1}{(k-1)/2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{(k-1)/2}}}} = \frac{\frac{k^3+3k}{2}}{\frac{k^2+1}{2}},$$

the proof follows. If k is even, then the period length is even by Theorem 5 and therefore $x^2 - dy^2 = -1$ has no positive integer solutions by Lemma 2. \square

Corollary 2. *Let $k > 1$ and $d = k^2 + 4$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is*

$$x_1 + y_1\sqrt{d} = \begin{cases} \frac{k^2+2}{2} + \frac{k}{2}\sqrt{d} & \text{if } k \text{ is even,} \\ \left(\frac{k^3+3k}{2} + \frac{k^2+1}{2}\sqrt{d}\right)^2 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. If k is even, then the proof follows from Lemma 2 and Theorem 5. If k is odd, then the proof follows from Corollary 1 and Lemma 1. \square

From Lemma 2 and Theorem 6, we can give the following corollary.

Corollary 3. *Let $k > 3$ and $d = k^2 - 4$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is given by*

$$x_1 + y_1\sqrt{d} = \begin{cases} \frac{k^2-2}{2} + \frac{k}{2}\sqrt{d} & \text{if } k \text{ is even,} \\ \frac{k^3-3k}{2} + \frac{k^2-1}{2}\sqrt{d} & \text{if } k \text{ is odd.} \end{cases}$$

Corollary 4. *Let $k > 3$. Then the equation $x^2 - (k^2 - 4)y^2 = -1$ has no integer solutions.*

Proof. The period length of continued fraction expansion of $\sqrt{k^2 - 4}$ is always even by Theorem 6. Thus by Lemma 2, it follows that there is no positive integer solutions of the equation $x^2 - (k^2 - 4)y^2 = -1$. \square

3. MAIN THEOREMS

Theorem 7. *Let $k > 1$ and $d = k^2 + 4$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by*

$$(x, y) = \begin{cases} \left(\frac{V_{2n}(k,1)}{2}, \frac{U_{2n}(k,1)}{2} \right) & \text{if } k \text{ is even,} \\ \left(\frac{V_{6n}(k,1)}{2}, \frac{U_{6n}(k,1)}{2} \right) & \text{if } k \text{ is odd,} \end{cases}$$

with $n \geq 1$.

Proof. Assume that k is even. Then, by Corollary 2 and Theorem 1, all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by

$$x_n + y_n\sqrt{d} = \left(\frac{k^2+2}{2} + \frac{k}{2}\sqrt{d} \right)^n$$

with $n \geq 1$. Let $\alpha_1 = \frac{k^2+2}{2} + \frac{k}{2}\sqrt{d}$ and $\beta_1 = \frac{k^2+2}{2} - \frac{k}{2}\sqrt{d}$. Then

$$x_n + y_n\sqrt{d} = \alpha_1^n \text{ and } x_n - y_n\sqrt{d} = \beta_1^n.$$

Thus it follows that $x_n = \frac{\alpha_1^n + \beta_1^n}{2}$ and $y_n = \frac{\alpha_1^n - \beta_1^n}{2\sqrt{d}}$. Let

$$\alpha = \frac{k + \sqrt{k^2 + 4}}{2} \text{ and } \beta = \frac{k - \sqrt{k^2 + 4}}{2}.$$

Then it is seen that $\alpha^2 = \alpha_1$ and $\beta^2 = \beta_1$. Thus it follows that

$$x_n = \frac{\alpha^{2n} + \beta^{2n}}{2} = \frac{V_{2n}(k, 1)}{2}$$

and

$$y_n = \frac{\alpha^{2n} - \beta^{2n}}{2\sqrt{d}} = \frac{\alpha^{2n} - \beta^{2n}}{2(\alpha - \beta)} = \frac{U_{2n}(k, 1)}{2}$$

by (1.1). Now assume that k is odd. Then by Corollary 2 and Theorem 1, we get

$$x_n + y_n\sqrt{d} = \left(\left(\frac{k^3 + 3k}{2} + \frac{k^2 + 1}{2}\sqrt{d} \right)^2 \right)^n$$

with $n \geq 1$. Let

$$\alpha_1 = \left(\frac{k^3 + 3k}{2} + \frac{k^2 + 1}{2}\sqrt{d} \right)^2$$

and

$$\beta_1 = \left(\frac{k^3 + 3k}{2} - \frac{k^2 + 1}{2}\sqrt{d} \right)^2.$$

Then

$$x_n + y_n\sqrt{d} = \alpha_1^n \text{ and } x_n - y_n\sqrt{d} = \beta_1^n.$$

Thus it is seen that $x_n = \frac{\alpha_1^n + \beta_1^n}{2}$ and $y_n = \frac{\alpha_1^n - \beta_1^n}{2\sqrt{d}}$. Let

$$\alpha = \frac{k + \sqrt{k^2 + 4}}{2} \text{ and } \beta = \frac{k - \sqrt{k^2 + 4}}{2}.$$

Since $\alpha_1 = \left(\frac{k^3 + 3k}{2} + \frac{k^2 + 1}{2}\sqrt{d} \right)^2 = (\alpha^3)^2 = \alpha^6$ and thus $\beta_1 = \beta^6$, we get

$$x_n = \frac{\alpha^{6n} + \beta^{6n}}{2} = \frac{V_{6n}(k, 1)}{2}$$

and

$$y_n = \frac{\alpha^{6n} - \beta^{6n}}{2\sqrt{d}} = \frac{\alpha^{6n} - \beta^{6n}}{2(\alpha - \beta)} = \frac{U_{6n}(k, 1)}{2}$$

by (1.1). Then the proof follows. \square

Theorem 8. *Let $k > 1$ be an odd integer and $d = k^2 + 4$. Then all positive integer solutions of the equation $x^2 - dy^2 = -1$ are given by*

$$(x, y) = \left(\frac{V_{6n-3}(k, 1)}{2}, \frac{U_{6n-3}(k, 1)}{2} \right)$$

with $n \geq 1$.

Proof. Assume that $k > 1$ be an odd integer. Then by Corollary 1 and Theorem 2, all positive integer solutions of the equation $x^2 - dy^2 = -1$ are given by

$$x_n + y_n\sqrt{d} = \left(\frac{k^3 + 3k}{2} + \frac{k^2 + 1}{2}\sqrt{d} \right)^{2n-1}$$

with $n \geq 1$. Let $\alpha_1 = \frac{k^3+3k}{2} + \frac{k^2+1}{2}\sqrt{d}$ and $\beta_1 = \frac{k^3+3k}{2} - \frac{k^2+1}{2}\sqrt{d}$. Then it follows that

$$x_n + y_n\sqrt{d} = \alpha_1^{2n-1} \text{ and } x_n - y_n\sqrt{d} = \beta_1^{2n-1}$$

and therefore $x_n = \frac{\alpha_1^{2n-1} + \beta_1^{2n-1}}{2}$ and $y_n = \frac{\alpha_1^{2n-1} - \beta_1^{2n-1}}{2\sqrt{d}}$. Let

$$\alpha = \frac{k + \sqrt{k^2 + 4}}{2} \text{ and } \beta = \frac{k - \sqrt{k^2 + 4}}{2}.$$

Then it is seen that

$$\alpha^3 = \left(\frac{k + \sqrt{k^2 + 4}}{2} \right)^3 = \frac{k^3 + 3k}{2} + \frac{k^2 + 1}{2}\sqrt{d} = \alpha_1$$

and

$$\beta^3 = \left(\frac{k - \sqrt{k^2 + 4}}{2} \right)^3 = \frac{k^3 + 3k}{2} - \frac{k^2 + 1}{2}\sqrt{d} = \beta_1.$$

Thus it follows that

$$x_n = \frac{(\alpha^3)^{2n-1} + (\beta^3)^{2n-1}}{2} = \frac{\alpha^{6n-3} + \beta^{6n-3}}{2} = \frac{V_{6n-3}(k, 1)}{2}$$

and

$$y_n = \frac{(\alpha^3)^{2n-1} - (\beta^3)^{2n-1}}{2\sqrt{d}} = \frac{\alpha^{6n-3} - \beta^{6n-3}}{2(\alpha - \beta)} = \frac{U_{6n-3}(k, 1)}{2}$$

by (1.1). Then the proof follows. \square

Theorem 9. Let $k > 3$ and $d = k^2 - 4$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by

$$(x, y) = \begin{cases} \left(\frac{V_{2n}(k, -1)}{2}, \frac{U_{2n}(k, -1)}{2} \right) & \text{if } k \text{ is even,} \\ \left(\frac{V_{3n}(k, -1)}{2}, \frac{U_{3n}(k, -1)}{2} \right) & \text{if } k \text{ is odd,} \end{cases}$$

with $n \geq 1$.

Proof. Assume that k is even. By Corollary 3 and Theorem 1, all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by

$$x_n + y_n\sqrt{d} = \left(\frac{k^2 - 2}{2} + \frac{k}{2}\sqrt{d} \right)^n.$$

Let $\alpha_1 = \frac{k^2 - 2}{2} + \frac{k}{2}\sqrt{d}$ and $\beta_1 = \frac{k^2 - 2}{2} - \frac{k}{2}\sqrt{d}$. Then it follows that

$$x_n + y_n\sqrt{d} = \alpha_1^n \text{ and } x_n - y_n\sqrt{d} = \beta_1^n.$$

and therefore $x_n = \frac{\alpha_1^n + \beta_1^n}{2}$ and $y_n = \frac{\alpha_1^n - \beta_1^n}{2\sqrt{d}}$. Let

$$\alpha = \frac{k + \sqrt{k^2 - 4}}{2} \text{ and } \beta = \frac{k - \sqrt{k^2 - 4}}{2}.$$

Then it is seen that $\alpha^2 = \alpha_1$ and $\beta^2 = \beta_1$. Thus it follows that

$$x_n = \frac{\alpha^{2n} + \beta^{2n}}{2} = \frac{V_{2n}(k, -1)}{2}$$

and

$$y_n = \frac{\alpha^{2n} - \beta^{2n}}{2\sqrt{d}} = \frac{\alpha^{2n} - \beta^{2n}}{2(\alpha - \beta)} = \frac{U_{2n}(k, -1)}{2}$$

by (1.2). Now assume that k is odd. Then by Corollary 3 and Theorem 1, we get

$$x_n + y_n\sqrt{d} = \left(\frac{k^3 - 3k}{2} + \frac{k^2 - 1}{2}\sqrt{d} \right)^n.$$

Let $\alpha_1 = \frac{k^3 - 3k}{2} + \frac{k^2 - 1}{2}\sqrt{d}$ and $\beta_1 = \frac{k^3 - 3k}{2} - \frac{k^2 - 1}{2}\sqrt{d}$. Then $x_n + y_n\sqrt{d} = \alpha_1^n$ and $x_n - y_n\sqrt{d} = \beta_1^n$. Thus it follows that $x_n = \frac{\alpha_1^n + \beta_1^n}{2}$ and $y_n = \frac{\alpha_1^n - \beta_1^n}{2\sqrt{d}}$. Let $\alpha = \frac{k + \sqrt{k^2 - 4}}{2}$ and $\beta = \frac{k - \sqrt{k^2 - 4}}{2}$. Since

$$\alpha^3 = \left(\frac{k + \sqrt{k^2 - 4}}{2} \right)^3 = \frac{k^3 - 3k}{2} + \frac{k^2 - 1}{2}\sqrt{d} = \alpha_1$$

and

$$\beta^3 = \left(\frac{k - \sqrt{k^2 - 4}}{2} \right)^3 = \frac{k^3 - 3k}{2} - \frac{k^2 - 1}{2}\sqrt{d} = \beta_1,$$

we get

$$x_n = \frac{\alpha^{3n} + \beta^{3n}}{2} = \frac{V_{3n}(k, -1)}{2}$$

and

$$y_n = \frac{\alpha^{3n} - \beta^{3n}}{2\sqrt{d}} = \frac{\alpha^{3n} - \beta^{3n}}{2(\alpha - \beta)} = \frac{U_{3n}(k, -1)}{2}$$

by (1.2). □

Now we give all positive integer solutions of the equations $x^2 - (k^2 + 4)y^2 = \pm 4$ and $x^2 - (k^2 - 4)y^2 = \pm 4$. Before giving all solutions of the equations $x^2 - (k^2 + 4)y^2 = \pm 4$, we give the following lemma which will be useful for finding the solutions.

Lemma 3. *Let $a + b\sqrt{d}$ be a positive integer solution to equation $x^2 - dy^2 = 4$. If $a > b^2 - 2$, then $a + b\sqrt{d}$ is the fundamental solution to the equation $x^2 - dy^2 = 4$.*

Proof. If $b = 1$, then the proof is trivial. Assume that $b > 1$. Suppose that $x_1 + y_1\sqrt{d}$ is a positive solution to the equation $x^2 - dy^2 = 4$ such that $1 \leq y_1 < b$. Then it follows that $a^2 - db^2 = 4 = x_1^2 - dy_1^2$ and thus $d = (x_1^2 - 4)/y_1^2 = (a^2 - 4)/b^2$. This shows that $x_1^2 b^2 - y_1^2 a^2 = 4b^2 - 4y_1^2 = 4(b^2 - y_1^2) > 0$. Thus

$$[(x_1 b + y_1 a)/2][(x_1 b - y_1 a)/2] = b^2 - y_1^2 > 1.$$

It can be seen that $x_1 b + y_1 a$ and $x_1 b - y_1 a$ are even integers. Let $(x_1 b + y_1 a)/2 = k_1$ and $(x_1 b - y_1 a)/2 = k_2$. Then $k_1 k_2 = b^2 - y_1^2$ and $a = (k_1 - k_2)/y_1$. Thus

$$a = \frac{k_1 - k_2}{y_1} \leq \frac{k_1 k_2 - 1}{y_1} = \frac{b^2 - y_1^2 - 1}{y_1} \leq b^2 - y_1^2 - 1 \leq b^2 - 2,$$

which is a contradiction since $a > b^2 - 2$. Then the proof follows. \square

Theorem 10. *Let $k > 1$. Then all positive integer solutions of the equation $x^2 - (k^2 + 4)y^2 = 4$ are given by*

$$(x, y) = (V_{2n}(k, 1), U_{2n}(k, 1))$$

with $n \geq 1$.

Proof. Let $a = k^2 + 2$ and $b = k$. Then $a + b\sqrt{k^2 + 4}$ is a positive integer solution of the equation $x^2 - (k^2 + 4)y^2 = 4$. Since $a = k^2 + 2 > k^2 - 2 = b^2 - 2$, it follows that $k^2 + 2 + k\sqrt{k^2 + 4}$ is the fundamental solution of the equation $x^2 - (k^2 + 4)y^2 = 4$, by Lemma 3. Thus, by Theorem 3, all positive integer solutions of the equation $x^2 - dy^2 = 4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(k^2 + 2 + k\sqrt{k^2 + 4})^n}{2^{n-1}} = 2 \left(\frac{k^2 + 2 + k\sqrt{k^2 + 4}}{2} \right)^n.$$

Let $\alpha_1 = \frac{k^2 + 2 + k\sqrt{k^2 + 4}}{2}$ and $\beta_1 = \frac{k^2 + 2 - k\sqrt{k^2 + 4}}{2}$. Then it is seen that

$$x_n + y_n\sqrt{d} = 2\alpha_1^n \text{ and } x_n - y_n\sqrt{d} = 2\beta_1^n.$$

Thus it follows that $x_n = \alpha_1^n + \beta_1^n$ and $y_n = \frac{\alpha_1^n - \beta_1^n}{\sqrt{d}}$. Let

$$\alpha = \frac{k + \sqrt{k^2 + 4}}{2} \text{ and } \beta = \frac{k - \sqrt{k^2 + 4}}{2}.$$

POSITIVE INTEGER SOLUTIONS OF THE PELL EQUATION $x^2 - dy^2 = N$, $d \in \{k^2 \pm 4, k^2 \pm 1\}$ AND $N \in \{\pm 1, \pm 4\}$

Then

$$\alpha^2 = \left(\frac{k + \sqrt{k^2 + 4}}{2} \right)^2 = \frac{k^2 + 2 + k\sqrt{k^2 + 4}}{2} = \alpha_1$$

and

$$\beta^2 = \left(\frac{k - \sqrt{k^2 + 4}}{2} \right)^2 = \beta_1.$$

Therefore we get

$$x_n = \alpha^{2n} + \beta^{2n} = V_{2n}(k, 1) \text{ and } y_n = \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{d}} = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = U_{2n}(k, 1)$$

by (1.1). Then the proof follows. \square

Theorem 11. *Let $k > 1$. Then all positive integer solutions of the equation $x^2 - (k^2 + 4)y^2 = -4$ are given by*

$$(x, y) = (V_{2n-1}(k, 1), U_{2n-1}(k, 1))$$

with $n \geq 1$.

Proof. Since $k^2 - (k^2 + 4) = -4$, it follows that $k + \sqrt{k^2 + 4}$ is the fundamental solution of the equation $x^2 - (k^2 + 4)y^2 = -4$. Thus by Theorem 4, all positive integer solutions of the equation $x^2 - dy^2 = -4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(k + \sqrt{k^2 + 4})^{2n-1}}{4^{n-1}} = 2 \left(\frac{k + \sqrt{k^2 + 4}}{2} \right)^{2n-1}.$$

Let $\alpha = \frac{k + \sqrt{k^2 + 4}}{2}$ and $\beta = \frac{k - \sqrt{k^2 + 4}}{2}$. Then it follows that

$$x_n + y_n\sqrt{d} = 2\alpha^{2n-1} \text{ and } x_n - y_n\sqrt{d} = 2\beta^{2n-1}.$$

Therefore

$$x_n = \alpha^{2n-1} + \beta^{2n-1} = V_{2n-1}(k, 1)$$

and

$$y_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{\sqrt{d}} = \frac{\alpha^{2n-1} - \beta^{2n-1}}{\alpha - \beta} = U_{2n-1}(k, 1)$$

by (1.1). Then the proof follows. \square

Theorem 12. *Let $k > 3$. Then all positive integer solutions of the equation $x^2 - (k^2 - 4)y^2 = 4$ are given by*

$$(x, y) = (V_n(k, -1), U_n(k, -1))$$

with $n \geq 1$.

Proof. Since $k^2 - (k^2 - 4) = 4$, it is seen that $k + \sqrt{k^2 - 4}$ is the fundamental solution of the equation $x^2 - (k^2 - 4)y^2 = 4$. Thus, by Theorem 3, all positive integer solutions of the equation $x^2 - dy^2 = 4$ are given by

$$x_n + y_n\sqrt{d} = \frac{(k + \sqrt{k^2 - 4})^n}{2^{n-1}} = 2 \left(\frac{k + \sqrt{k^2 - 4}}{2} \right)^n.$$

Let $\alpha = \frac{k + \sqrt{k^2 - 4}}{2}$ and $\beta = \frac{k - \sqrt{k^2 - 4}}{2}$. Then it follows that $x_n + y_n\sqrt{d} = 2\alpha^n$ and $x_n - y_n\sqrt{d} = 2\beta^n$. Thus we get

$$x_n = \alpha^n + \beta^n = V_n(k, -1) \text{ and } y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}} = \frac{\alpha^n - \beta^n}{\alpha - \beta} = U_n(k, -1)$$

by (1.2). □

The following theorem is given in [5].

Theorem 13. *Let d be odd positive integer. If the equation $x^2 - dy^2 = -4$ has positive integer solution, then the equation $x^2 - dy^2 = -1$ has positive integer solutions.*

Now we give the continued fraction expansions of $\sqrt{k^2 + 1}$ and $\sqrt{k^2 - 1}$. Since the continued fraction expansions of them are given in [3], we omit their proofs.

Theorem 14. *Let $k \geq 1$. Then $\sqrt{k^2 + 1} = [k, 2k]$ and if $k > 1$, then $\sqrt{k^2 - 1} = [k - 1, 1, 2(k - 1)]$.*

The proofs of the following corollaries follow from Lemma 2 and Theorem 14 and therefore we omit their proofs.

Corollary 5. *Let $k \geq 1$ and $d = k^2 + 1$. Then the fundamental solution of the equation $x^2 - dy^2 = 1$ is*

$$x_1 + y_1\sqrt{d} = 2k^2 + 1 + 2k\sqrt{d}.$$

Corollary 6. *Let $k \geq 1$ and $d = k^2 + 1$. Then the fundamental solution of the equation $x^2 - dy^2 = -1$ is*

$$x_1 + y_1\sqrt{d} = k + \sqrt{d}.$$

Corollary 7. *Let $k > 1$ and $d = k^2 - 1$. Then the fundamental solution of the equation $x^2 - dy^2 = 1$ is*

$$x_1 + y_1\sqrt{d} = k + \sqrt{d}.$$

Theorem 15. *Let $k \geq 1$. Then all positive integer solutions of the equation $x^2 - (k^2 + 1)y^2 = 1$ are given by*

$$(x, y) = \left(\frac{V_{2n}(2k, 1)}{2}, U_{2n}(2k, 1) \right)$$

with $n \geq 1$.

Proof. By Corollary 5 and Lemma 2, it follows that all positive integer solutions of the equation $x^2 - (k^2 + 1)y^2 = 1$ are given by

$$x_n + y_n \sqrt{k^2 + 1} = \left(2k^2 + 1 + 2k \sqrt{k^2 + 1} \right)^n = \left(2k^2 + 1 + k \sqrt{(2k)^2 + 4} \right)^n.$$

Let $\alpha = \frac{2k + \sqrt{(2k)^2 + 4}}{2}$ and $\beta = \frac{2k - \sqrt{(2k)^2 + 4}}{2}$. Then

$$\alpha^2 = \left(\frac{2k + \sqrt{(2k)^2 + 4}}{2} \right)^2 = 2k^2 + 1 + k \sqrt{(2k)^2 + 4}$$

and

$$\beta^2 = \left(\frac{2k - \sqrt{(2k)^2 + 4}}{2} \right)^2 = 2k^2 + 1 - k \sqrt{(2k)^2 + 4}.$$

Thus it follows that

$$x_n + y_n \sqrt{k^2 + 1} = x_n + \frac{y_n}{2} \sqrt{(2k)^2 + 4} = \alpha^{2n}$$

and

$$x_n - y_n \sqrt{k^2 + 1} = x_n - \frac{y_n}{2} \sqrt{(2k)^2 + 4} = \beta^{2n}.$$

Then it is seen that

$$x_n = \frac{\alpha^{2n} + \beta^{2n}}{2} = \frac{V_{2n}(2k, 1)}{2}$$

and

$$y_n = \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{(2k)^2 + 4}} = \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = U_{2n}(2k, 1)$$

by (1.1). □

Since the proof of the following theorems are similar to that of above theorems, we omit them.

Theorem 16. *Let $k \geq 1$. Then all positive integer solutions of the equation $x^2 - (k^2 + 1)y^2 = -1$ are given by*

$$(x, y) = \left(\frac{V_{2n-1}(2k, 1)}{2}, U_{2n-1}(2k, 1) \right)$$

with $n \geq 1$.

Theorem 17. *Let $k > 1$. Then all positive integer solutions of the equation $x^2 - (k^2 - 1)y^2 = 1$ are given by*

$$(x, y) = \left(\frac{V_n(2k, -1)}{2}, U_n(2k, -1) \right)$$

with $n \geq 1$.

Corollary 8. *Let $k > 1$. Then the equation $x^2 - (k^2 - 1)y^2 = -1$ has no positive integer solutions.*

Proof. The period length of continued fraction expansion of $\sqrt{k^2 - 1}$ is always even by Theorem 14. Thus, by Lemma 2, it follows that there is no positive integer solutions of the equation $x^2 - (k^2 - 1)y^2 = -1$. \square

Theorem 18. *Let $k > 3$. Then the equation $x^2 - (k^2 - 4)y^2 = -4$ has no positive integer solutions.*

Proof. Assume that k is odd. Then $k^2 - 4$ is odd and thus the proof follows from Theorem 13 and Corollary 4. Now assume that k is even. If (a, b) is a solution to the equation $x^2 - (k^2 - 4)y^2 = -4$, then a is even. Thus we get

$$(a/2)^2 - ((k/2)^2 - 1)b^2 = -1,$$

which is impossible by Corollary 8. Then the proof follows. \square

Now we give all positive integer solutions of the equations $x^2 - (k^2 + 1)y^2 = \pm 4$ and $x^2 - (k^2 - 1)y^2 = \pm 4$.

Theorem 19. *Let $k \geq 1$ and $k \neq 2$. Then all positive integer solutions of the equation $x^2 - (k^2 + 1)y^2 = -4$ are given by*

$$(x, y) = (V_{2n-1}(2k, 1), 2U_{2n-1}(2k, 1))$$

with $n \geq 1$.

Proof. Since $k \geq 1$, it can be shown that $2k + 2\sqrt{k^2 + 1}$ is the fundamental solution to the equation $x^2 - (k^2 + 1)y^2 = -4$. Then by Theorem 4, all positive integer solutions of the equation $x^2 - (k^2 + 1)y^2 = -4$ are given by

$$x_n + y_n \sqrt{k^2 + 1} = 2 \left(\frac{2k + 2\sqrt{k^2 + 1}}{2} \right)^{2n-1} = 2 \left(\frac{2k + \sqrt{(2k)^2 + 4}}{2} \right)^{2n-1}.$$

Let $\alpha = \frac{2k + \sqrt{(2k)^2 + 4}}{2}$ and $\beta = \frac{2k - \sqrt{(2k)^2 + 4}}{2}$. Then we get

$$x_n + y_n \sqrt{k^2 + 1} = x_n + \frac{y_n}{2} \sqrt{(2k)^2 + 4} = 2\alpha^{2n-1}$$

and

$$x_n - y_n \sqrt{k^2 + 1} = x_n - \frac{y_n}{2} \sqrt{(2k)^2 + 4} = 2\beta^{2n-1}.$$

Thus it follows that

$$x_n = \alpha^{2n-1} + \beta^{2n-1} = V_{2n-1}(2k, 1)$$

and

$$y_n = 2 \frac{\alpha^{2n-1} - \beta^{2n-1}}{\sqrt{(2k)^2 + 4}} = 2 \frac{\alpha^{2n-1} - \beta^{2n-1}}{\alpha - \beta} = 2U_{2n-1}(2k, 1)$$

by (1.1). \square

Now we can give the following corollary from Theorem 19 and identity (1.4).

Corollary 9. *If (a, b) is a positive integer solution of the equation $x^2 - (k^2 + 1)y^2 = -4$, then a and b are even.*

Since the proof of the following theorem is similar to that of Theorem 19, we omit it.

Theorem 20. *Let $k > 1$. Then all positive integer solutions of the equation $x^2 - (k^2 - 1)y^2 = 4$ are given by*

$$(x, y) = (V_n(2k, -1), 2U_n(2k, -1))$$

with $n \geq 1$.

Theorem 21. *Let $k \geq 1$ and $k \neq 2$. Then all positive integer solutions of the equation $x^2 - (k^2 + 1)y^2 = 4$ are given by*

$$(x, y) = (V_{2n}(2k, 1), 2U_{2n}(2k, 1))$$

with $n \geq 1$.

Proof. Firstly, we show that if (a, b) is a solution to the equation $x^2 - (k^2 + 1)y^2 = -4$, then a and b are even. Assume that k is odd. Then $k^2 + 1 = 2t$ for some odd integer t . Since $a^2 - 2tb^2 = 4$, it follows that a is even and therefore b is even. Now assume that k is even. Let $d = k^2 + 1$. Then d is odd. Assume that a and b are odd integers. Let $x_1 = |db - ka|$, $y_1 = |a - kb|$. Then x_1 and y_1 are odd integers. Moreover,

$$x_1^2 - dy_1^2 = (db - ka)^2 - d(a - kb)^2 = b^2d(d - k^2) + a^2(k^2 - d) = b^2d - a^2 = -(a^2 - db^2) = -4$$

Thus $x_1 + y_1\sqrt{d}$ is a positive solution of the equation $x^2 - (k^2 + 1)y^2 = -4$, which is impossible by Corollary 9. Therefore if $a + b\sqrt{d}$ is any solutions of the equation $x^2 - dy^2 = 4$, then a and b are even integers and thus $\frac{a}{2} + \frac{b}{2}\sqrt{d}$ is a solution to the equation $x^2 - dy^2 = 1$. Then it follows that the fundamental solution of the equation $x^2 - dy^2 = 4$ is $4k^2 + 2 + 4k\sqrt{d}$, by corollary 5. Thus

by Theorem 3, it follows that all positive integer solutions of the equation $x^2 - (k^2 + 1)y^2 = 4$ are given by

$$x_n + y_n \sqrt{k^2 + 1} = 2 \left(\frac{4k^2 + 2 + 4k\sqrt{k^2 + 1}}{2} \right)^n = 2 \left(\frac{4k^2 + 2 + 2k\sqrt{(2k)^2 + 4}}{2} \right)^n.$$

Let $\alpha = \frac{2k + \sqrt{(2k)^2 + 4}}{2}$ and $\beta = \frac{2k - \sqrt{(2k)^2 + 4}}{2}$. Then

$$\alpha^2 = \left(\frac{2k + \sqrt{(2k)^2 + 4}}{2} \right)^2 = \frac{4k^2 + 2 + 2k\sqrt{(2k)^2 + 4}}{2}$$

and

$$\beta^2 = \left(\frac{2k - \sqrt{(2k)^2 + 4}}{2} \right)^2 = \frac{4k^2 + 2 - 2k\sqrt{(2k)^2 + 4}}{2}.$$

Thus it follows that $x_n + y_n \sqrt{k^2 + 1} = x_n + \frac{y_n}{2} \sqrt{(2k)^2 + 4} = 2\alpha^{2n}$ and $x_n - \frac{y_n}{2} \sqrt{(2k)^2 + 4} = 2\beta^{2n}$. Then it is seen that

$$x_n = \alpha^{2n} + \beta^{2n} = V_{2n}(2k, 1)$$

and

$$y_n = 2 \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{(2k)^2 + 4}} = 2 \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = 2U_{2n}(2k, 1),$$

by (1.1). □

It can be shown that if $k > 2$, then the continued fraction expansion of $\sqrt{k^2 - k}$ is $[k - 1, 1, 2(k - 1)]$ (see [2], page 234). Therefore we can give the following corollary easily.

Corollary 10. *Let $k > 2$. Then the equation $x^2 - (k^2 - k)y^2 = -1$ has no positive integer solutions.*

Corollary 11. *Let $k \geq 2$ and $k \neq 3$. Then the equation $x^2 - (k^2 - 1)y^2 = -4$ has no positive integer solutions.*

Proof. Assume that k is even. Then $k^2 - 1$ is odd and the proof follows from Theorem 13 and Corollary 8.

Assume that k is odd and . Then $k^2 - 1$ is even. Now assume that $a^2 - (k^2 - 1)b^2 = -4$ for some positive integers a and b . Then a is even and this implies that

$$(a/2)^2 - [(k^2 - 1)/4]b^2 = -1.$$

This is impossible by Corollary 10, since

$$(k^2 - 1)/4 = ((k + 1)/2)^2 - (k + 1)/2.$$

□

Continued fraction expansion of $\sqrt{5}$ is $[2, \overline{4}]$. Then the period length of the continued fraction expansion of $\sqrt{5}$ is 1. Therefore the fundamental solution to the equation $x^2 - 5y^2 = 1$ is $9 + 4\sqrt{5}$ and the fundamental solution to the equation $x^2 - 5y^2 = -1$ is $2 + \sqrt{5}$ by Lemma 2. Therefore, by using (1.3), we can give the following corollaries easily.

Corollary 12. *All positive integer solutions of the equation $x^2 - 5y^2 = 1$ are given by*

$$(x, y) = \left(\frac{L_{6n}}{2}, \frac{F_{6n}}{2} \right)$$

with $n \geq 1$.

Corollary 13. *All positive integer solutions of the equation $x^2 - 5y^2 = -1$ are given by*

$$(x, y) = \left(\frac{L_{6n-3}}{2}, \frac{F_{6n-3}}{2} \right)$$

with $n \geq 1$.

It can be seen that fundamental solutions of the equations $x^2 - 5y^2 = -4$ and $x^2 - 5y^2 = 4$ are $1 + \sqrt{5}$ and $3 + \sqrt{5}$, respectively. Thus we can give following corollaries.

Corollary 14. *All positive integer solutions of the equation $x^2 - 5y^2 = 4$ are given by*

$$(x, y) = (L_{2n}, F_{2n})$$

with $n \geq 1$.

Corollary 15. *All positive integer solutions of the equation $x^2 - 5y^2 = -4$ are given by*

$$(x, y) = (L_{2n-1}, F_{2n-1})$$

with $n \geq 1$.

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SAKARYA UNIVERSITY, MATHEMATICS DEPARTMENT, TR 54187 SAKARYA, TURKEY
E-mail address: rkeskin@sakarya.edu.tr

SAKARYA UNIVERSITY, MATHEMATICS DEPARTMENT, TR 54187 SAKARYA, TURKEY
E-mail address: mer_ney_mat@hotmail.com